Journal of Engineering Mathematics **47:** 161–174, 2003.

Radiation of water waves from an oscillating source located at a slope of angle $\pi/4$

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Received 18 March 2002; accepted in revised form 3 June 2003

Abstract. An analytical investigation is performed of the linear radiation problem for water waves generated by an oscillating normal velocity distribution along a sloping beach with slope angle $\pi/4$. The distribution of normal velocity is arbitrary, and it oscillates with a given frequency. The solution is expressed in terms of the Green function which represents a source of unit oscillatory flux located at an arbitrary position along the slope. At infinity the radiation condition is applied to determine the outgoing wave. As a simple example of integrating the Green function, the reflection of an incoming sinusoidal wave is calculated.

Key words: Green function, oscillating source, radiation, slope, water waves

1. Introduction

The present work is concerned with the linear theory of water waves generated by an oscillating normal-velocity distribution along a sloping beach. Earthquake disturbances may be oscillatory. An important question is whether the wave energy generated by local oscillatory disturbances on a sloping beach will be trapped locally or radiate out into the open ocean.

The present model assumes two-dimensional incompressible potential flow. For mathematical convenience, we consider only a slope angle of $\pi/4$. The same problem has been considered by Otay and Kazezyilmaz-Alhan [1]. They found no energy radiation at infinity, which is in conflict with our results. We will express the solution in terms of the Green function for an oscillating wall source.

A more general version of our Green-function problem has been investigated by Sretenskii [2]. He considered a sloping beach of angle $\pi/2n$ (where n is a positive integer) and a submerged oscillating source at an arbitrary position. Our problem is the special case where $n = 2$ and the source is located on the slope. Sretenskii concentrated on finding the wave-free positions of the source. There are a number of continuous curves extending from the surface to the slope, connecting the source positions for which there is no outward radiation of waves at infinity.

Morris [3] studied a submerged oscillating source with a beach of arbitrary slope angle between 0 and π. The theory was not elaborated far enough to develop any explicit formula for the outgoing waves, which will be done in our less general model. For the wave-free source positions, Morris [4] found results in agreement with Sretenskii's [2]. In the case of overhanging cliffs (slope angle between $\pi/2$ and π), qualitatively new features appeared. The wave-free loci would be closed loops from one surface point to another one.

The present problem is related to the linearized version of edge waves on beaches, which has often been treated by shallow-water approximations. We will follow Minzoni and Whitham

[5] in that we apply the full Laplace equation for water waves. Our solution is related to the solution for an oscillating wall source at a slope angle $\pi/2$ (*i.e.*, vertical wall). This is given by Wehausen and Laitone ([6, pp. 479–482]). The wall source for a vertical wall is equivalent to the submerged oscillatory source, as long as we only consider the flux from the right-hand side of the source. We then introduce a fictitious vertical wall through the submerged source.

Submerged oscillating sources satisfying the free-surface condition are important in the theory of water waves. Source distributions are useful for solving linear diffraction and radiation problems for various body shapes.

2. Problem formulation

We consider inviscid, irrotational flow in a homogeneous and incompressible fluid. The flow is governed by the velocity potential $\Phi(x, y, t)$. The x-axis is directed along the undisturbed free surface and the y-axis is vertical. The fluid domain is confined by a uniform slope of angle $\pi/4$; see the definition sketch in Figure 1. The coordinate system is centered on the waterline so that the fluid domain is given by $x > 0, \; -x < y < 0$. The gravitational acceleration is denoted by g.

We will study the time-periodic free-surface flow due to a wall source oscillating with a given angular frequency ω. The oscillatory part is separated out:

$$\Phi(x, y, t) = \mathrm{e}^{\mathrm{j}\omega t}\phi(x, y). \tag{1}$$

The time-independent potential $\phi(x, y)$ will be complex, and the real part of the total potential Φ has physical significance. Here j is the imaginary unit. The normal velocity distribution along the beach slope is generally given by a function $f(s)$.

$$\frac{\partial\phi}{\partial n} = f(s) = \int_0^\infty f(s_0)\delta(s - s_0)\mathrm{d}s_0,\; y = -x, (x > 0). \tag{2}$$

Here n is a coordinate normal to the beach and pointing into the fluid domain. The coordinate $s = x\sqrt{2}$ points downward along the slope. The waterline is represented by $s = 0$. We have introduced Dirac's delta function δ. The source contribution at $s = s_0$ in the integrand corresponds to the Green function of the present problem. The normal velocity condition for the oscillating Green function is defined by

$$\frac{\partial\phi}{\partial n} = q_0\,\delta(s - s_0),\; y = -x \; (x > 0). \tag{3}$$

The local source strength for the normal derivative is $q_0 = f(s_0)$. Integration with respect to s_0 recovers the original normal velocity distribution (2).

We introduce the depth H of the source. Since the slope angle is $\pi/4$, the oscillating source is located in the point $(x, y) = (H, -H)$, and $s_0 = \sqrt{2}H$. We introduce a parameter K representing the wavenumber at infinity (as $x \to \infty$):

$$K = \frac{\omega^2}{g}. \tag{4}$$

The linearized free-surface condition is

$$\frac{\partial\phi}{\partial y} - K\phi = 0,\; y = 0,\; (x > 0). \tag{5}$$

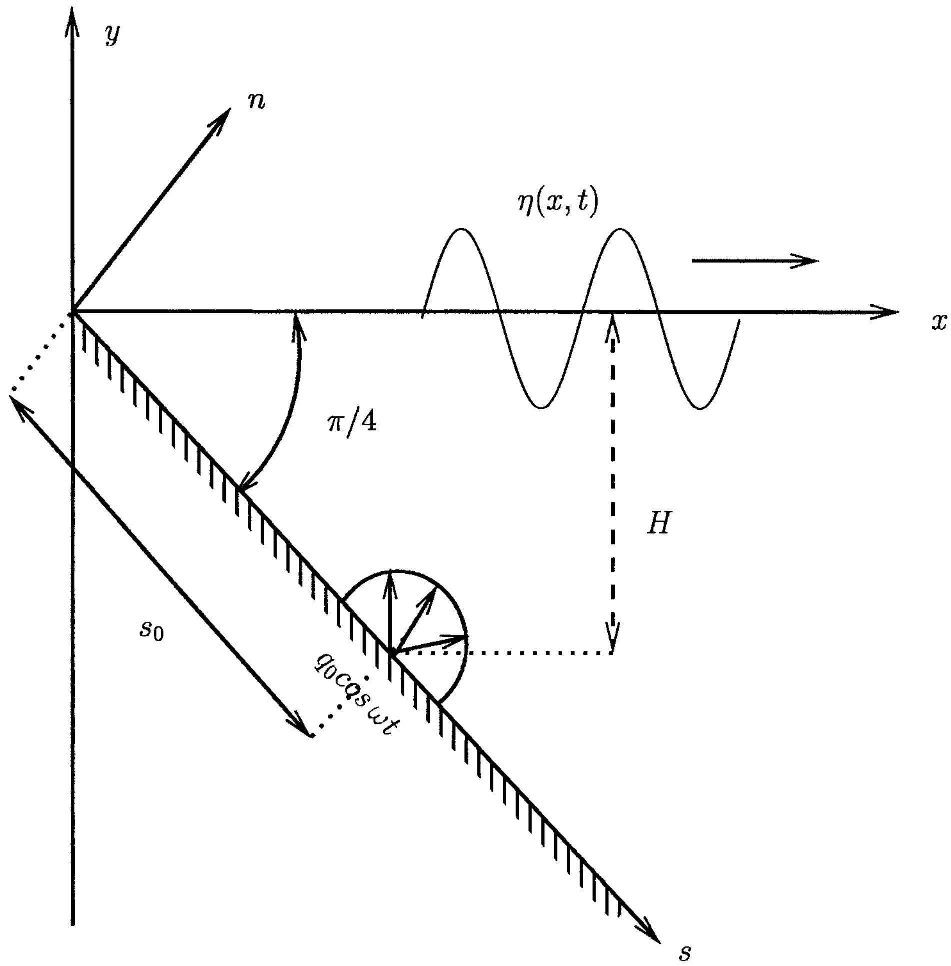

Figure 1. A definition sketch of a line source of pulsating flux $q = q_0 \cos \omega t$ located on the bottom of an impenetrable beach of slope angle $\pi/4$.

The radiation condition is

$$\frac{\partial \phi}{\partial x} + \mathrm{j}K\phi = 0,\ x \to \infty\ (y = 0). \tag{6}$$

The disturbance must vanish in the limit of infinite depth:

$$|\nabla \phi| \to 0,\quad y \to -\infty. \tag{7}$$

The surface elevation can be given in two different ways:

$$\eta(x, t) = -\frac{1}{g}\left.\frac{\partial \Phi}{\partial t}\right|_{y=0} = -\frac{\omega}{g}\Re\mathfrak{e}\left(\mathrm{j}\,\mathrm{e}^{\mathrm{j}\omega t}\phi(x, 0)\right) = -\frac{1}{\omega}\,\Re\mathfrak{e}\left(\mathrm{j}\,\mathrm{e}^{\mathrm{j}\omega t}\left.\frac{\partial \phi}{\partial y}\right|_{y=0}\right). \tag{8}$$

3. Solution procedure

The potential is split into two contributions

$$\phi = \phi_0 + \phi_\omega, \tag{9}$$

where ϕ_0 is the locally pulsating potential, taking care of the source singularity and obeying the equipotential condition at $y = 0$; ϕ_ω is the non-singular potential of the propagating wave. By the image method we find the singular source potential ϕ_0.

$$\phi_0 = \frac{q_0}{\pi} \log \frac{z^2 + 2\mathrm{i}H^2}{z^2 - 2\mathrm{i}H^2} \tag{10}$$

The complex variable z is defined by $z = x + \mathrm{i}y$, where i is the associated imaginary unit. The source strength q_0 for a boundary source is twice as large as the source strength for a submerged source with the same flux.

We will work with dimensionless variables in the solution procedure. The transformation to dimensionless variables is achieved formally by putting $q_0 = 1$ and $H = 1$, and redefining K as the dimensionless parameter $\omega^2 H/g$. With these redefinitions, the dimensionless boundary-value problem will be the same as the one with dimension. The dimensionless version of the pulsating potential ϕ_0 is

$$\phi_0 = \frac{1}{\pi} \log \frac{z^2 + 2\mathrm{i}}{z^2 - 2\mathrm{i}}. \tag{11}$$

The free-surface condition (5) can now be written as

$$\frac{\partial \phi_\omega}{\partial y} - K\phi_\omega = -\eta_0, \ \ y = 0 \ (x > 0). \tag{12}$$

where we introduce

$$\eta_0 = \left.\frac{\partial \phi_0}{\partial y}\right|_{y=0} = \frac{8x}{\pi(x^4 + 4)}. \tag{13}$$

By the method of separation of variables we write a Fourier component of ϕ_ω as $X(x)Y(y)$. From Laplace's equation we find

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \pm k^2. \tag{14}$$

We will integrate over the wavenumber k in order to generate Fourier-integral solutions. The wall condition (3) implies that:

$$\frac{\partial \phi_\omega}{\partial n} = 0, \ \ y = -x \ (x > 0). \tag{15}$$

This wall condition (15) suggests an image method with Fourier components that are invariant under the transformation $(x, y) \to (-y, -x)$. There are only two classes of Fourier components that satisfy both this condition and the condition of spatial decay at infinite depth (7):

$$\phi_{\omega 1} = \mathrm{e}^{ky} \cos kx + \mathrm{e}^{-kx} \cos ky, \tag{16}$$

$$\phi_{\omega 2} = \mathrm{e}^{ky} \sin kx - \mathrm{e}^{-kx} \sin ky. \tag{17}$$

In order to construct these modes, it was necessary to combine the two classes of contributions involving plus and minus signs in the right-hand side of (14) for each value of k. This was demonstrated by Havelock [7] in his classical theory of the oscillating wavemaker. The general solution will be a Fourier integral over these two classes.

$$\phi_\omega = \int_0^\infty \left(A(k)(e^{ky}\ \cos\ kx + e^{-kx}\ \cos\ ky) + B(k)(e^{ky}\ \sin\ kx - e^{-kx}\ \sin\ ky)\right)\ dk. \tag{18}$$

Following Wehausen and Laitone [6, pp. 479–482] and Palm [8] we find that a first-order pole at $k = K$ is necessary to generate an outgoing wave. This is because a common factor $(k - K)$ produced by the free-surface condition (12) cancels the singularity at the free surface. In the present problem, this cancellation happens only when $A(k) + B(k) = 0$. The Fourier integral (18) reduces to

$$\phi_\omega = \int_0^\infty A(k)\left(e^{ky}\ \cos\ kx + e^{-kx} \cos\ ky - e^{ky} \sin\ kx + e^{-kx} \sin\ ky\right)\ dk, \tag{19}$$

which is inserted into the free-surface condition (12):

$$\int_0^\infty (k - K)A(k)\left(\cos\ kx - \sin\ kx + e^{-kx}\right)\ dk = -\eta_0. \tag{20}$$

We apply the Fourier-Sine transform [9, p. 167] and find

$$\eta_0 = \frac{8x}{\pi(x^4 + 4)} = \frac{4}{\pi}\int_0^\infty e^{-k}\ \sin\ k\ \sin\ kx\ dk. \tag{21}$$

The original Fourier coefficients $A(k)$ are replaced by modified ones $C(k)$:

$$C(k) = \frac{\pi}{4}(k - K)A(k). \tag{22}$$

The integral equation for $C(k)$ is

$$\int_0^\infty C(k)\left(\cos\ kx - \sin\ kx + e^{-kx}\right)\ dk = -\frac{2x}{x^4 + 4}. \tag{23}$$

Equation (21) indicates that we should search for solutions for $C(k)$ of the type $e^{-k}\ \sin\ k$, which will produce an antisymmetric extension around $x = 0$ through evaluating the Fourier-Sine integral (21). However, the left-hand side of the modified Fourier integral (23) does not have the antisymmetric form of a pure sine. We need a broader class of solutions with both symmetric and antisymmetric terms, and are lead to attempt a solution of the following type:

$$C(k) = (a \cos k + b \sin k)\ e^{-k}. \tag{24}$$

The integral on the left-hand side of (23) for coefficients of the form (24) can be evaluated according to Rottmann [9, p. 167].

$$\frac{4a + 4b - 4bx}{x^4 + 4} = -\frac{2x}{x^4 + 4}. \tag{25}$$

This determines the parameters $a = -1/2$ and $b = 1/2$. The attempted solution (24) is indeed the full solution. Equation (22) gives the original Fourier coefficients $A(k)$:

$$A(k) = \frac{2}{\pi}\ \frac{\sin\ k - \cos\ k}{k - K}\ e^{-k}. \tag{26}$$

Our way of determining $C(k)$ in (23) may seem like a lucky guess. It is a short version of an iteration process that is explained in the Appendix. A second method based on the Mellin transform is also described in the Appendix.

4. The radiated wave

The potential (19) must be integrated in the complex k-plane so that the radiation condition (6) is satisfied. In the far-field $x \to \infty$ we can neglect the terms e^{-kx}. The propagating potential in the far-field can be split into two integrals.

$$\phi_\omega = I_1 + I_2 \tag{27}$$

$$= \frac{1+j}{\pi}\int_0^\infty \frac{(\sin k - \cos k)e^{k(y-1)}e^{jkx}}{k-K}\,dk + \frac{1-j}{\pi}\int_0^\infty \frac{(\sin k - \cos k)e^{k(y-1)}e^{-jkx}}{k-K}\,dk.$$

The question is how to deform the integration path around the singularity $k = K$ in order to satisfy (6). This radiation condition requires that the terms with x-dependence of the type e^{jKx} must vanish in the far-field $x \to \infty$. The first integral I_1 must evaluate to zero, which again requires that its pole at $k = K$ is not circumvented. Since I_1 diverges for $\Im\mathfrak{m}(k) < 0$, the integration path has to be closed in the upper half plane. Then the limiting path of integration along the x-axis must be deformed above the pole to avoid a contribution from I_1. Note that all contributions where $\Im\mathfrak{m}(k) > 0$ will vanish in I_1 in the limit $x \to \infty$, to guarantee that there will not be any contribution from points outside the real axis. Then the value of I_1 evaluated along the real axis is zero, provided that the path is deformed above the singularity.

We found that the integration path in the k-plane for the integral I_1 must be deformed above the singularity. The choice of an integration path must apply to the total solution and therefore also to the second integral I_2. Its integrand diverges for $\Im\mathfrak{m}(k) > 0$ in the far-field ($x \to \infty$), while it vanishes for all $\Im\mathfrak{m}(k) < 0$. This means that the closed integral will circumvent the pole at $k = K$. Moreover this implies that the value of the integral along the real axis will be equal to the contribution from the pole at $k = K$. From the residue theorem with negative direction of circulation it follows that

$$\lim_{x\to\infty} \phi_\omega = 2(j+1)(\cos K - \sin K)e^{K(y-1)}e^{-jKx}. \tag{28}$$

We reintroduce dimensional variables by the transformations:

$$(x, y, K) \to \left(\frac{x}{H}, \frac{y}{H}, KH\right). \tag{29}$$

Moreover, we replace the unit source strength by the flux amplitude q_0 and reintroduce the time factor $e^{j\omega t}$. The full time-dependent potential in the far-field will then be:

$$\lim_{x\to\infty} \Phi(x, y, t) = 2q_0(j+1)(\cos KH - \sin KH)e^{K(y-H)}e^{j(\omega t - Kx)}. \tag{30}$$

We apply the definition (4) and evaluate the surface elevation at infinity by (8).

$$\lim_{x\to\infty} \eta(x, t) = \frac{4\omega\, q_0}{g}\, e^{-\frac{H\omega^2}{g}} \cos\left(\frac{H\omega^2}{g} + \frac{\pi}{4}\right) \cos\left(\frac{\omega^2}{g}x - \omega t + \frac{\pi}{4}\right). \tag{31}$$

There is outward radiation, except at the discrete angular frequencies ω_m given by

$$\omega_m = \sqrt{\frac{\pi g}{H}\left(m - \frac{3}{4}\right)}, \quad m = 1, 2, \ldots. \tag{32}$$

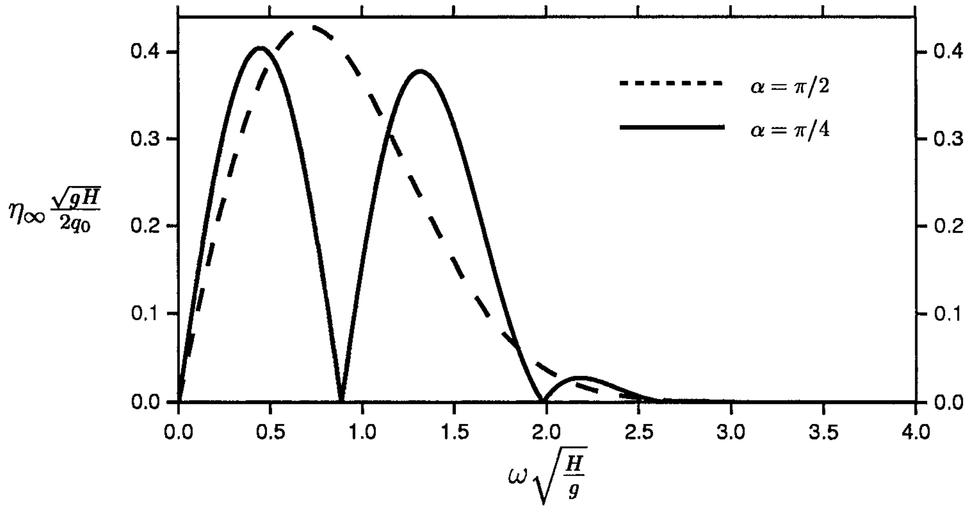

Figure 2. The amplitude of the outgoing wave plotted against source frequency. The dashed graph represents the amplitude when the slope angle is $\pi/2$ and is included for comparison.

This simple formula for wave-free frequencies when the source is located at a slope of angle $\pi/4$ is known from Sretenskii [2] and Morris [4]. Figure 2 shows how the amplitude of the outgoing wave varies with the angular frequency of the oscillating source.

The standing surface oscillation can be calculated from (8) and (10).

$$\eta_0(x,t) = \frac{8q_0}{\pi\omega}\frac{H^2 x}{x^4 + 4H^4}\sin\omega t. \tag{33}$$

This pulsating surface elevation varies as $\sin\omega t$. The integral of the surface velocity from $x = 0$ to $x = \infty$ is $q_0 \cos\omega t$. Due to mass balance, the incompressible surface flow is indeed forced by the source strength $q_0\cos\omega t$. The maximum oscillation amplitude takes place at $x_0 = (4/3)^{1/4}H$ and is given by $0{\cdot}51307\ q_0/(\omega H)$. We have not calculated the total elevation for finite values of x when there is a travelling wave. The ratio between the amplitude of the far-field radiated wave and the maximum amplitude of the standing oscillation is given by

$$\frac{4\pi\sqrt{2}}{3^{3/4}}KH\mathrm{e}^{-KH}\cos\left(KH + \frac{\pi}{4}\right).$$

and is plotted against angular frequency in Figure 3. The maximum absolute value of this ratio is 1·98915 and occurs at $KH = 1{\cdot}9113$. This corresponds to an angular frequency $\omega = 1{\cdot}3825\sqrt{g/H}$, which will be a slow oscillation of order 0·1 radians per second for typical ocean depths. For high-frequency oscillations, the amplitude of the local standing oscillations will dominate over the amplitude of the radiated wave.

Following Mei [10, p. 17], the outward energy flux per unit width from the oscillating source is given by:

$$c_g E = 4\rho\omega q_0^2 \mathrm{e}^{-2\omega^2 H/g}\cos^2(\omega^2 H/g + \pi/4) \tag{34}$$

Here the group velocity c_g is given by $c_g = \omega/(2K) = g/(2\omega)$ and E denotes the wave-energy density per unit width from the oscillating slope source.

We will now consider the energy flux more closely in the case of a normal velocity amplitude U_0 being constant over a small portion of the slope of length L. This means that

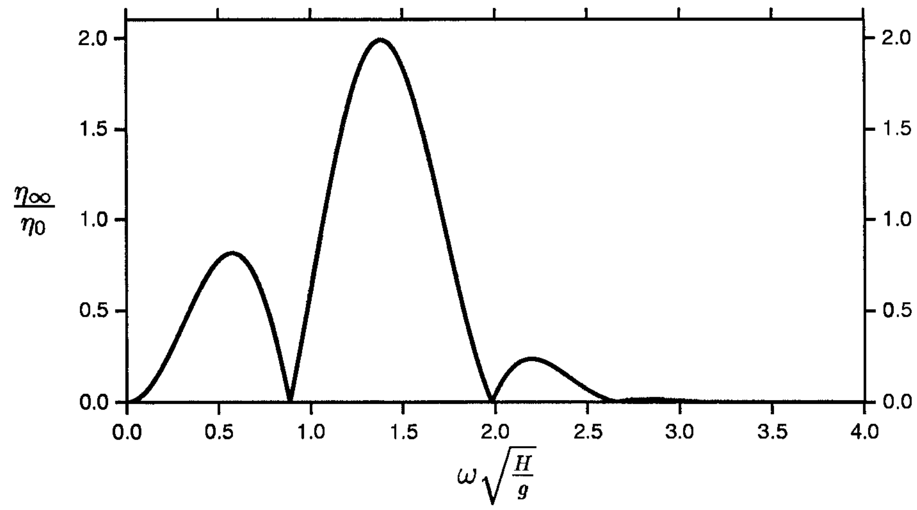

Figure 3. The ratio of the amplitude of the outgoing wave to the amplitude of the locally pulsating surface elevation plotted against frequency.

$q_0 = U_0 L$, and the normal velocity of the strip of length L is $U_0 \cos \omega t$. The deflection amplitude D_0 of the bottom is $D_0 = U_0/\omega$, and its time dependence is sin ωt. Let us now insert $q_0 = L D_0 \omega$ in the formula for the energy flux and assume the deflection amplitude D_0 to be constant instead of the source flux amplitude q_0. Then the expression (34) can be rewritten

$$c_g E = 4\rho\omega^3 L^2 D_0^2 \mathrm{e}^{-2\omega^2 H/g} \cos^2(\omega^2 H/g + \pi/4). \tag{35}$$

The maximum energy flux per unit width occurs for the angular frequency $\omega = 1{\cdot}351\sqrt{g/H}$. Plots of how the energy flux varies with angular frequency, both in the case of a fixed flux amplitude and in the case of a fixed deflection amplitude, are included in Figure 4.

Let us compare the radiated wave from a source on a slope with the similar solution for an oscillating source in a vertical wall. Following Wehausen and Laitone [6, pp. 479–482] we find the radiated wave at infinity from an oscillating wall source (subscript: wall) located in the point $(x, y) = (0, -H)$ at slope angle $\pi/2$ and with flux q_0.

$$\lim_{x\to\infty} \eta_{\mathrm{wall}}(x,t) = \frac{2\omega\, q_0}{g}\, \mathrm{e}^{-\frac{H\omega^2}{g}} \cos\left(\frac{\omega^2}{g}x - \omega t\right). \tag{36}$$

The source at the vertical wall radiates energy for all finite frequencies, in contrast to the source at the slope (Equation (31)). The standing surface oscillation due to an oscillating source at a vertical wall is

$$\eta_{0,\mathrm{wall}}(x,t) = \frac{2q_0}{\pi\omega} \frac{\sin\omega t}{x^2 + H^2}. \tag{37}$$

We want to compare the wave from a slope source with that from a wall source. We take the ratio between their time-averaged squared amplitudes at infinity:

$$\frac{\lim_{x\to\infty} \overline{\eta^2}}{\lim_{x\to\infty} \overline{\eta^2_{\mathrm{wall}}}} = 4\cos^2\left(\frac{H\omega^2}{g} + \frac{\pi}{4}\right). \tag{38}$$

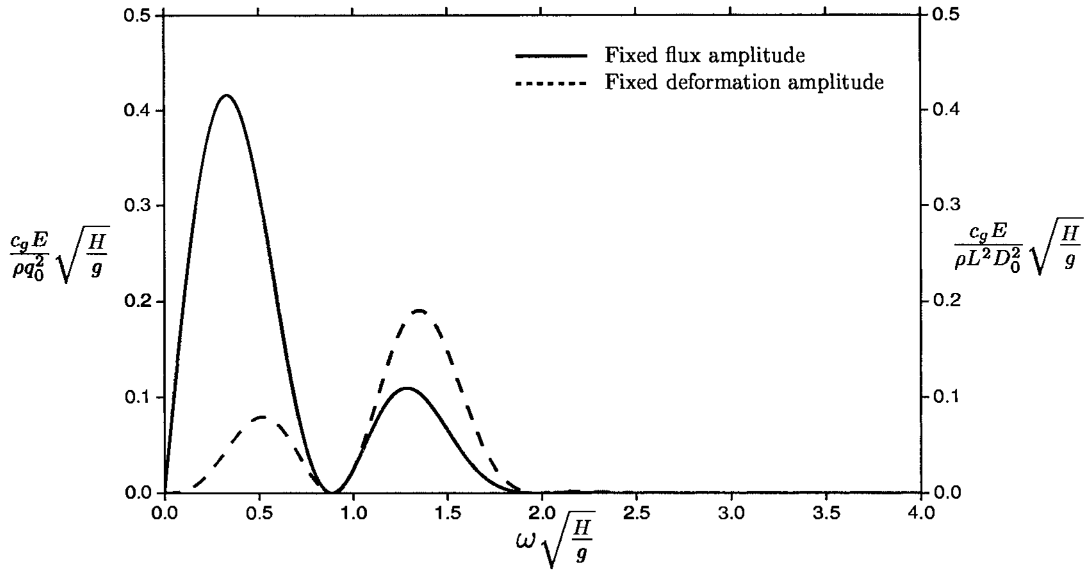

Figure 4. The outgoing energy flux $c_g E$ plotted against frequency for two different scenarios. The solid curve corresponds to keeping the flux amplitude of the source fixed, while the dashed curve corresponds to fixing the deflection amplitude instead. The left-hand axis applies to the solid curve and the right-hand axis applies to the dashed curve.

The over-bar denotes the time-average. Equation (38) represents the ratio between the energy flux from the two different sources. The outgoing wave from the slope source is usually stronger than the wave from the wall source, except for frequency intervals around the wave-free ones for the slope source. In this comparison between a slope source and a wall source, we have assumed that they have the same depth below the free surface, the same frequency and the same flux amplitude.

The phase of the outgoing wave is not the same for the slope source and a wall source located at the same depth. There are two reasonable ways to compare their phase shifts.

(i) The first option for comparing the wave phase from the two different sources is to assume the waterline position to be fixed (at $x = 0$). This is the choice we have made in our definition of the coordinate system. A comparison between (31) and (36) shows that the outgoing wave from the wall source is always shifted an angle $\alpha = \pi/4$ ahead of the outgoing wave from the slope source. This corresponds to a shift of one eighth of a wavelength, valid for any frequency.

(ii) The second option for comparing the wave phase from the two types of source is to let the source position be the same in both cases. For the source at the slope, this means that the waterline position must be redefined as $x = -H$. The result is that the outgoing wave from the wall source is shifted ahead of the outgoing wave from the slope source by an angle $\alpha = \pi/4 + \omega^2 H/g$. This gives a phase shift that increases monotonically with the square of the frequency. It is remarkable that, each time the phase shift angle α is equal to $(n + 1/2)\pi$ (where n is an integer), there is no wave radiation. All values for the phase-shift angle are possible, except for the values $\pi/2$ and $-\pi/2$ when there is no wave emitted from the slope source. In terms of spatial shift, the crests of the outgoing wave from the slope source cannot be shifted a quarter of a wavelength apart from the crests of the wave from a wall source at the same position. But all other spatial shifts between the outgoing wave trains from the two different sources are possible, as both wave trains will have nonzero amplitudes.

5. Conclusions

In the present paper a linearized problem of water-wave radiation has been investigated. A closed-form analytical solution is found for the water-wave radiated out from an oscillating source located at a slope of angle $\pi/4$. This exact result gives an extension of the previous work by Sretenskii [2] and Morris [3, 4] on more general versions of the same problem. They determined the waveless frequencies in our problem, which are included in our general formula for the elevation of the outgoing wave. The longest wavelength at which there is no outward radiation is eight times the submergence depth of the oscillating source at the slope.

The present work gives the Green function for an oscillating normal velocity along a slope. By integrating up the Green function, we are able to construct the outgoing wave for any distribution of normal velocity along the slope, assuming the frequency is given.

A basic example of integrating the Green function is an incoming wave of amplitude ϵ:

$$\eta(x,t) = \epsilon\ \cos(Kx + \omega t). \tag{39}$$

The requirement of zero normal velocity along the slope implies a complex form of the normal velocity distribution for the oscillating source

$$f(s) = \frac{\omega\epsilon}{\sqrt{2}}\,(1 - \mathrm{j})\ \mathrm{e}^{(\mathrm{j}-1)Ks/\sqrt{2}}. \tag{40}$$

We replace q_0 by $f(s)$ in (30), substitute $H = s/\sqrt{2}$ and integrate from $s = 0$ to $s = \infty$. The final result for the total wave in the far-field is

$$\lim_{x\to\infty} \eta_{\text{total}}(x,t) = 2\epsilon \cos\left(Kx + \frac{\pi}{4}\right)\ \cos\left(\omega t - \frac{\pi}{4}\right). \tag{41}$$

This gives full reflection of the incoming wave power, which is a requirement of our inviscid model. A nontrivial result for the far-field standing wave (41) is the location of its node points, where a phase shift angle of $\pi/4$ appears. One way to interpret this phase shift is to compare it with the node points that would result from the same incoming wave being reflected from a vertical wall. The reflected wave from the slope of angle $\pi/4$ with waterline at $x = 0$ is the same as a reflected wave from a solid vertical wall located in $x = -\lambda/8$, where $\lambda = 2\pi/K$ is the wavelength of the incoming wave.

The standing oscillation represented by η_0 (Equation (33)) can also be integrated, and produces a non-singular solution of the type described by Stoker [11]. He also identified another possible mathematical solution, giving a standing oscillation that is singular at the waterline. According to the present work, Stoker's singular solution does not have anything to do with dissipation of energy, as there will be full reflection and no dissipation according to the exact linearized problem. Observations show a significant loss of mechanical energy in waves of small amplitude incident on a sloping beach, but this must be caused by viscous shear stress at the slope. The dissipation in the tongue of water near the waterline will be strong, especially because of the contact line singularity caused by the no-slip condition at the slope.

The advantage of our Green-function solution for the reflection problem compared with the classical solutions of standing oscillations, is that the radiation condition is applied. Thus, we solve the exact reflection problem for the slope. Surprisingly, this seems not to have been done in the classical literature on this established problem. The search for standing oscillations at the slope is not equivalent to solving the reflection problem. Our linearized solution for the reflection of inviscid water waves from a sloping beach is consistent, as it gives full reflection of the incoming energy flux.

Acknowledgment

We thank the referees for very useful comments.

Appendix. Solution of an integral equation

Our task is to solve the integral equation (23):

$$\int_0^\infty C(k)\left(\cos kx - \sin kx + \mathrm{e}^{-kx}\right)\,\mathrm{d}k = -\frac{2x}{x^4+4}. \tag{A.1}$$

We will demonstrate two independent methods of solution.

AN ITERATIVE APPROACH

We will first demonstrate an iteration method for finding the Fourier coefficients $C(k)$. The integral suggests that we can express the right-hand side in terms of a combination of a Fourier-Cosine, a Fourier-Sine and a Laplace transform. Our iteration technique assumes that we can express $C(k)$ as a series in which each individual term is a transform of one of the above mentioned kinds:

$$C(k) = \sum_{i=0}^{n} C_i(k). \tag{A.2}$$

We find the first term $C_1(k)$ of the series by letting one of the integrals satisfy the right-hand side of the integral equation (A.1) independently. Incidentally, the right-hand side is an odd function, so we choose $C_1(k)$ to be equal to its Fourier-Sine transform:

$$C_1(k) = -\int_0^\infty \frac{2x}{x^4+4}\sin kx\,\mathrm{d}x = -\frac{\pi}{2}\mathrm{e}^{-k}\sin k. \tag{A.3}$$

The evaluation of this integral can be done by the inverse transform of (21) in the main text.

If the kernel of the integral equation (A.1) had contained only the odd term sin kx, the solution would have been of the type $\mathrm{e}^{-k}\sin k$. Since the kernel contains two other terms, this is only a first iteration to the solution. We need to evaluate the integrals produced by this attempted solution $C_1(k)$ from each of the three terms in the kernel, following Rottmann [9, p. 167]:

$$I_1 = \int_0^\infty \mathrm{e}^{-k}\sin k\ \cos kx\,\mathrm{d}k = \frac{2-x^2}{x^4+4},\ I_2 = \int_0^\infty \mathrm{e}^{-k}\sin k\ \sin kx\,\mathrm{d}k = \frac{2x}{x^4+4},$$

$$I_3 = \int_0^\infty \mathrm{e}^{-k}\sin k\ \mathrm{e}^{-kx}\,\mathrm{d}k = \frac{x^2-2x+2}{x^4+4}.$$

When inserted into (A.1), $C_1(k)$ produces the following right-hand side:

$$-\frac{\pi}{2}\int_0^\infty \mathrm{e}^{-k}\sin k\Big(\cos kx-\sin kx+\mathrm{e}^{-kx}\Big)\mathrm{d}k=-\frac{\pi}{2}(I_1-I_2+I_3)=2\pi\frac{x-1}{x^4+4}.$$

Here the second term is an even function and encourages us to set the next term of the series (A.2) equal to its Fourier-Cosine transform. According to Rottmann [9, p. 167], this evaluates to:

$$C_2(k)=-\int_0^\infty \frac{1}{x^4+4}\cos\,kx\,\mathrm{d}x=-\frac{\pi}{8}\mathrm{e}^{-k}\Big(\cos k+\sin\,k\Big). \tag{A.4}$$

Note that this iteration technique involves an iteration with respect to functions, not amplitudes. Therefore we have chosen to disregard the factor 2π in C_2. Once the right function is found, we can adjust its amplitude because the integral equation (A.1) is linear.

Now there are three new integrals produced from the three kernel terms when the second iteration is inserted into the integral equation (A.1); see Rottmann [9, p. 167]:

$$I_4=\int_0^\infty \mathrm{e}^{-k}\cos\,k\,\cos\,kx\,\mathrm{d}k=\frac{2+x^2}{x^4+4},\ I_5=\int_0^\infty \mathrm{e}^{-k}\cos\,k\,\sin\,kx\,\mathrm{d}k=\frac{x^3}{x^4+4},$$

$$I_6=\int_0^\infty \mathrm{e}^{-k}\cos\,k\,\mathrm{e}^{-kx}\,\mathrm{d}k=\frac{(1+x)(x^2-2x+2)}{x^4+4}.$$

When inserted into (A.1), $C_2(k)$ produces the following right-hand side:

$$-\frac{\pi}{8}\int_0^\infty \mathrm{e}^{-k}\Big(\cos k+\sin k\Big)\Big(\cos kx-\sin kx+\mathrm{e}^{-kx}\Big)\mathrm{d}k=\frac{\pi}{2}\frac{x}{x^4+4}-\pi\frac{1}{x^4+4}.$$

Here we have utilized all the integrals $I_1, I_2, \ldots, I_6$. Upon substitution into (A.1) we see that $C_1(k)$ and $C_2(k)$ result in two different linear combinations of the desired odd function $x/(x^4+1)$ and the unwanted even function $1/(x^4+4)$. Since the integral equation (A.1) is linear, we need to find a linear combination of $C_1(k)$ and $C_2(k)$ for which this even function cancels. The simplest combination achieving this is $C_1(k)-2C_2(k)$, which produces a term $\pi x/(x^4+4)$ upon substitution in the integral equation. We need to multiply this right-hand side by $-2/\pi$ to get the right expression for $C(k)$. In fact this implies that

$$-\frac{2}{\pi}(C_1(k)-2C_2(k))=C(k). \tag{A.5}$$

Finally this gives us the solution

$$C(k)=\frac{1}{2}\mathrm{e}^{-k}\Big(\sin k-\cos k\Big). \tag{A.6}$$

A MELLIN-TRANSFORM APPROACH

Let us start from the beginning with a different approach for solving the integral equation (23). A systematic procedure based on the Mellin transform can be given for solving the following class of integral equations [14]:

$$\int_0^\infty C(k)G(kx)\mathrm{d}k=F(x). \tag{A.7}$$

Our Equation (A.1) belongs to this class. Its known functions are:

$$G(x) = \cos x - \sin x + \mathrm{e}^{-x},\ F(x) = -\frac{2x}{x^4+4}.$$

The Mellin transform of a function $f(x)$ is defined as

$$\tilde{f}(s) = \int_0^\infty f(x)\, x^{s-1}\mathrm{d}x, \tag{A.8}$$

where s is defined within a strip of the complex plane ($\alpha < \mathfrak{Re}(s) < \beta$). The inverse Mellin transform is given by

$$f(x) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \tilde{f}(s)\, x^{-s}\mathrm{d}s, \tag{A.9}$$

where the inversion integral is placed in a strip $\alpha < c < \beta$.

The tilde ($\tilde{\ }$) superscript will denote transformed quantities, being functions of the complex variable s. The Mellin transforms of the known functions are found in Oberhettinger [15, p. 25, 42]

$$\tilde{G}(s) = \Gamma(s)\left(\cos\frac{\pi s}{2} - \sin\frac{\pi s}{2} + 1\right),\quad (0 < \mathfrak{Re}(s) < 1),$$

and in Rottmann [9, p. 154]

$$\tilde{F}(s) = -\pi\sqrt{2}^{s-5}\csc\frac{\pi(1+s)}{4},\quad (-1 < \mathfrak{Re}(s) < 3).$$

Titchmarsh [12, p. 315] has shown that the Mellin transform of the unknown function $C(k)$ for the class of Equations (A.7) is given by

$$\tilde{C}(s) = \frac{\tilde{F}(1-s)}{\tilde{G}(1-s)}.$$

We implement the variable change $s \to 1-s$ in the known transformed functions. We then perform standard trigonometric manipulations by MATHEMATICA. Finally, we apply the identity given in Rottmann [9, p. 109]:

$$\Gamma(1-s)\Gamma(s)\sin\pi s = \pi.$$

After these manipulations, we arrive at the Mellin transform of the unknown Fourier coefficients:

$$\tilde{C}(s) = 2^{-\frac{s}{2}-1}\Gamma(s)\left(\sin\frac{\pi s}{4} - \cos\frac{\pi s}{4}\right). \tag{A.10}$$

In [13, p. 191] the inverse Mellin transforms of $\Gamma(s)\sin as$ and $\Gamma(s)\cos as$ are given. An extra factor $2^{-s/2}$ is easily incorporated in the inversion integral (A.9) by replacing the variable x by $\sqrt{2}k$. By inserting $a = \pi/4$ we finally arrive at the formula

$$C(k) = \frac{1}{2}\mathrm{e}^{-k}\Big(\sin k - \cos k\Big), \tag{A.11}$$

which confirms the result (A.6) obtained above by the iteration method above.

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